

5-cycles and the Petersen graph

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Abstract

We show that if G is a connected bridgeless cubic graph whose every 2-factor is comprised of cycles of length five then G is the Petersen graph.

"The Petersen graph is an obstruction to many properties in graph theory, and often is, or conjectured to be, the only obstruction". This phrase is taken from one of the series of papers by Robertson, Sanders, Seymour and Thomas that is devoted to the proof of prominent Tutte conjecture- a conjecture which states that if the Petersen graph is not a minor of a bridgeless cubic graph G then G is 3-edge-colorable, and which in its turn is a particular case of a much more general conjecture of Tutte stating that every bridgeless graph G has a nowhere zero 4-flow unless the Petersen graph is not a minor of G .

Another result that stresses the exceptional role of the Petersen graph is proved by Alspach et al. in [1]. The following striking conjecture of Jaeger states that everything related to the colorings of bridgeless cubic graphs can be reduced to that of the Petersen graph, more specifically,

Conjecture 1 *Petersen coloring conjecture of Jaeger [4]: the edges of every bridgeless cubic graph G can be mapped into the edges of the Petersen graph in such a way that any three mutually incident edges of G are mapped to three mutually incident edges of the Petersen graph.*

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The Petersen graph is so important in graph theory that even an entire book is written about it [2].

It is known that for every bridgeless cubic graph G and its two edges e and f we can always find a 2-factor of G containing these two edges. Recently, Jackson and Yoshimoto in [3] observed that in any bridgeless cubic graph without multiple edges we can always find a triangle-free 2-factor. An earlier result by Rosenfeld [7] also worths to be mentioned here, which states that there are infinitely many 3-connected cubic graphs whose every 2-factor contains a cycle of length no more five.

Sometimes it is convinient to have a 2-factor F of a bridgeless cubic graph G such that not all cycles of F are 5-cycles (=cycles of length five) [6]. The main reason why we are interested in 5-cycles is the following: it is known that it is the odd cycles of a graph (particularly, a bridgeless cubic graph) that prevent it to have a 3-edge-coloring. Fortunately, the triangles can be overcome easily. This is due to the operation of the contraction of the triangles, which preserves the cubicness and bridgelessness of a graph. Therefore, we need a technique to cope with odd cycles of length at least five, and particularly, the 5-cycles. The main result of this paper states that unless G is the Petersen graph in every connected bridgeless cubic graph G we can always find a 2-factor F that contains a cycle which is not a 5-cycle.

We consider finite, undirected graphs without loops. Graphs may contain multiple edges. We follow [5, 9] for the terminology.

Theorem 2 (*Tutte, [8]*): *A graph G contains a perfect matching if and only if for every $S \subseteq V(G)$ $o(G - S) \leq |S|$, where $o(H)$ denotes the number of odd components of a graph H .*

Corollary 3 *If G is a bridgeless cubic graph, then for every its edge e there is a perfect matching F with $e \in F$.*

This corollary immediately implies

Corollary 4 *If G is a bridgeless cubic graph, then for every its edge e there is a perfect matching F with $e \notin F$.*

We will also need the following property of the Petersen graph:

Proposition 5 *The Petersen graph is the unique cubic graph of girth five on ten vertices.*

We are ready to state the main result of the paper:

Theorem 6 *If G is a connected bridgeless cubic graph whose every 2-factor is comprised of 5-cycles then G is the Petersen graph.*

Proof. First of all note that G is not 3-edge-colorable, thus every 2-factor of G contains at least two odd cycles.

Claim 7 G does not have a cycle of length two.

Proof. Suppose that G contains a cycle C of length two. Let u and v be the vertices of C , and let u', v' be the other ($\neq v, \neq u$) neighbours of u and v , respectively. Note that since G is bridgeless, we have $u' \neq v'$. Now let F be a perfect matching of G containing the edge (u, u') (corollary 3). Clearly, $(v, v') \in F$. Consider the complementary 2-factor of F . Note that C is a cycle in this 2-factor contradicting the condition of the theorem. ■

Claim 8 G does not have two triangles sharing an edge.

Proof. Let u, v, w and u', v, w be two triangles of G which share the edge (v, w) . Clearly, $(u, u') \notin E(G)$, as G is not 3-edge-colorable. Let u_1 and u'_1 be the other ($\neq v, w$) neighbours of u and u' , respectively. Note that since G is bridgeless we have $u_1 \neq u'_1$. Consider a perfect matching F with $(v, w) \in F$ (corollary 3). Clearly, $(u, u_1), (u_1, u'_1) \in F$. Note that the complementary 2-factor of F contains the 4-cycle on vertices u, v, u', w contradicting the condition of theorem. ■

Claim 9 G does not have a square and a triangle sharing an edge.

Proof. Suppose, on the contrary, that G contains a square $(u, v), (v, x), (x, w), (w, u)$ and a triangle $(v, y), (y, x), (x, v)$ which share the edge (x, v) . Due to claim 8, $(u, x) \notin E(G)$, $(v, w) \notin E(G)$. Now, let F be a perfect matching of G containing the edge (u, w) (corollary 3). Clearly, $(v, x) \in F$ and there is a vertex $y' \notin \{u, v, x, w, y\}$ such that $(y, y') \in E(G)$. Now, consider a path $u, (u, v), v, (v, y), y, (y, x), x, (x, w), w$ of length four. The path lies on a cycle C of the complementary 2-factor of F . Due to claim 7 there is only one edge connecting u and w . Thus the length of C is at least six contradicting the condition of theorem. ■

Claim 10 G does not have a triangle.

Proof. On the opposite assumption, consider a triangle C on vertices x, y, z of G . Since G is bridgeless we imply that there are vertices x', y', z' adjacent to x, y, z , respectively, that do not lie on C . Now, consider a perfect matching F of G containing the edge (x, y) (corollary 3). Clearly, $(z, z') \in F$. Note that the path $y', (y', y), y, (y, z), z, (z, x), x, (x, x'), x'$ of length four lies on some cycle C' of the complementary 2-factor of F . Claim 8 implies that $(y', x') \notin E(G)$ thus the length of C' is at least six contradicting the condition of theorem. ■

Claim 11 G does not have a square, and girth of G is five.

Proof. Assume G to contain a square $C = (u, v), (v, w), (w, z), (z, u)$. Claim 10 implies that $(u, w) \notin E(G)$, $(v, z) \notin E(G)$. Let u_1, v_1, w_1, z_1 be the vertices of G that are adjacent to u, v, w, z , respectively and do not lie on C . Consider a perfect matching F of G containing the edge (u, u_1) (corollary 3). Clearly,

$\{(v, v_1), (w, w_1), (z, z_1)\} \not\subseteq F$, as if it were true then the complementary 2-factor of F would have contained C as a cycle, which contradicts the condition of theorem. Thus

$$|F \cap \{(v, v_1), (w, w_1), (z, z_1)\}| = 1$$

Without loss of generality, we may assume that $(v, v_1) \in F$. Note that $(w, z) \in F$. Now, consider the cycle C_F in the complementary 2-factor of F , which contains the path $z_1, (z_1, z), z, (z, u), u, (u, v), v, (v, w), w, (w, w_1), w_1$. Due to claim 10 $z_1 \neq w_1$ thus the length of C_F is at least six contradicting the condition of theorem. Thus, G cannot contain a square, too, therefore its girth is five. ■

Claim 12 G is 3-edge-connected.

Proof. Suppose, for a contradiction, that G is only 2-edge-connected, and let $(u, v), (u', v')$ be two edges which form a 2-edge cut so that u and u' are in the same component of $G \setminus \{(u, v), (u', v')\}$. Now, there must exist a perfect matching not using (u, v) (corollary 4), so the complementary 2-factor must contain a 5-cycle which uses both the edges (u, v) and (u', v') . It follows that either $(u, u') \in E(G)$ or $(v, v') \in E(G)$. Without loss of generality, we may assume that $(v, v') \in E(G)$. Let w be the neighbor of v other than u, v' , and let w' be the neighbor of v' other than u', v . Now, there exists a perfect matching containing the edge (v, v') (corollary 3), and the complementary 2-factor must contain a 5-cycle which uses all of the edges $(u, v), (v, w), (u', v'), (v', w')$. It follows that either $u = u'$ or $v = v'$, but either of these contradicts the fact that G is bridgeless. This contradiction shows that G is 3-edge connected. ■

Claim 13 Every 3-edge-cut of G consists of three edges incident to a common vertex.

Proof. Let $(U, \bar{U}) = \{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ be a 3-cut of G and suppose that $\{u_1, u_2, u_3\} \subseteq U$, $\{v_1, v_2, v_3\} \subseteq \bar{U}$. We claim that either $u_1 = u_2 = u_3$ or $v_1 = v_2 = v_3$. Before showing this let us show that there is no edge connecting u_i and u_j or v_i and v_j , $1 \leq i < j \leq 3$. On the opposite assumption, suppose that $(v_1, v_2) \in E(G)$. Let v'_1 and v'_2 be the neighbours of v_1 and v_2 , respectively, that are different from u_1, v_2 and u_2, v_1 . Clearly, $v'_1, v'_2 \in \bar{U}$ and claim 10 implies that $v'_1 \neq v'_2$. Consider a perfect matching $F_{1,2}$ of G containing the edge (v_1, v_2) (corollary 3). Since $|U|$ is odd ((U, \bar{U}) is an odd cut), we have $(u_3, v_3) \in F_{1,2}$. Thus, the complementary 2-factor of $F_{1,2}$ must contain a 5-cycle containing the edges $(u_1, v_1), (v_1, v'_1), (u_2, v_2), (v_2, v'_2)$. Since $v'_1 \neq v'_2$ we have $u_1 = u_2$. Note that u_1, v_1, v_2 forms a triangle contradicting claim 10. Thus $(v_1, v_2) \notin E(G)$. Similarly, the absence of the other edges can be shown. Now, let us turn to the proof of claim 13. Let F be a perfect matching missing (u_1, v_1) (corollary 4). Since $|U|$ is odd, we imply that F contains one of $(u_2, v_2), (u_3, v_3)$ and misses the other one. Without loss of generality, we may assume that $(u_3, v_3) \in F$, $(u_2, v_2) \notin F$. Note that there should be a 5-cycle containing both the edges (u_2, v_2) and (u_3, v_3) . As $(u_1, u_2) \notin F$, $(v_1, v_2) \notin F$, we imply that either $u_1 = u_2$ or $v_1 = v_2$. Again, we can assume that $u_1 = u_2$. Let us show that $u_1 = u_3$, too.

Suppose that $u_1 \neq u_3$. Let w be a vertex from U adjacent to the vertex $u_1 = u_2$, and let $U' = U \setminus \{u_1\}$. Note that $(U', \bar{U}') = \{(u_1, w), (u_3, v_3)\}$ is a 2-edge-cut of G contradicting the choice of claim 12. Thus $u_1 = u_3$ and we are done. ■

Claim 14 *If $u, v, w, x \in V(G)$ and $(u, v), (v, w), (w, x) \in E(G)$ then there is a perfect matching of G containing both (u, v) and (w, x) .*

Proof. Suppose, for a contradiction, that the statement does not hold. Then $G' = G \setminus \{u, v, w, x\}$ has no perfect matching, so by theorem 2 there exists a subset of vertices $Y \subseteq V(G')$ so that $G' \setminus Y$ has more than $|Y|$ odd components. Let I be the set of isolated vertices in $G' \setminus Y$, let O be the set of odd components of $G' \setminus Y$ with at least three vertices, and let E be the set of even components of $G' \setminus Y$. We know that $|Y| < |I| + |O|$ by assumption, but in fact $|Y| + 2 \leq |I| + |O|$ since $|Y| - |I| - |O|$ must be an even number (as $|V(G')|$ is even). Now, let $Y^+ = Y \cup \{u, v, w, x\}$ and let C be the edge cut which separates Y^+ from $V(G) \setminus Y^+$. It follows from our construction that $|C| \leq 3|Y| + 6$ since every vertex in Y can contribute at most three edges to C , and there are at most six edges in C with one of u, v, w, x as endpoint. On the other hand, claim 13 implies that every component in $O \cup E$ must contribute at least four edges to C , and every vertex in I contributes exactly three edges to C , so

$$|C| \geq 3|I| + 4|O| + 4|E| \geq 3(|Y| + 2) + |O| + 4|E|$$

It follows from this that $O = E = \emptyset$, and that every vertex in Y must have all three incident edges in C . Thus $G \setminus \{(u, v), (v, w), (w, x)\}$ is a bipartite graph. Now, there exists a perfect matching of G which contains the edge (u, v) , and every odd cycle in the complementary 2-factor must contain (v, w) and (w, x) , so the complementary 2-factor cannot have two odd cycles - giving us a contradiction. ■

Now we are ready to complete the proof of the theorem. Claim 14 implies that every 3-edge path must be contained in a cycle of length five, and it follows from this that every 2-edge path of is contained in at least two 5-cycles. Let u be a vertex of G , let v, w, x be the neighbors of u , and assume that the neighbors of v, w, x are $\{u, v_1, v_2\}$, $\{u, w_1, w_2\}$, and $\{u, x_1, x_2\}$, respectively. It follows from the fact that G has girth five that all of these vertices we have named are distinct. Since the 2-edge path with edges $(v, u), (u, w)$ is contained in two cycles of length five, there must be at least two edges between $\{v_1, v_2\}$ and $\{w_1, w_2\}$. Similarly, there are at least two edges between $\{w_1, w_2\}$ and $\{x_1, x_2\}$, and between $\{x_1, x_2\}$ and $\{v_1, v_2\}$. As G is connected we imply that $V(G) = \{u, v, w, x, v_1, v_2, w_1, w_2, x_1, x_2\}$, and there are exactly two edges between $\{v_1, v_2\}$ and $\{w_1, w_2\}$, $\{w_1, w_2\}$ and $\{x_1, x_2\}$, $\{x_1, x_2\}$ and $\{v_1, v_2\}$. Proposition 5 implies that G is isomorphic to the Petersen graph. ■

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